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The ω-Functions, a Class of Normal Functions occurring in Statistics.

By E. Cunningham, M.A., Fellow of St. John's College, Cambridge, Lecturer in Applied Mathematics, University College, London.

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I.—*Introductory.*

The present paper originated in an attempt to discover the significance of certain functions developed by Professor Karl Pearson in a memoir entitled "A Mathematical Theory of Random Migration," and by him called *ω-functions.* They belong to the category of normal functions, and are applied in the memoir named to obtain an expansion to represent a distribution symmetrical about a point in a plane. The distribution is not fixed but depends on a parameter σ, the function ω_{2n} being a function of $x = r/\sigma$, r being the distance from the centre of the distribution. The fundamental differential equation is $\{d^2/dx^2 + (x+x^{-1})\,d/dx + 2\,(n+1)\}\,\omega_{2n}(x) = 0$.

In the course of the present investigation it soon appeared that the same function led to solutions of the equation of conduction of heat in two dimensions for the case of symmetry round the origin, the time t taking the place of σ^2. In fact, it was found that if a solution of that equation is sought in the form $f(t)\,\phi\,(r^2/t)$, that solution is $t^{-(n+1)}\,\omega_{2n}(r^2/4t)$, n being arbitrary. The function ω_{2n} is equal to $e^{-r^2/4t}$ multiplied by a polynomial, and is therefore especially adapted to the solution of the problem of the cooling of an infinite sheet, the temperature at a great distance being always zero.

The next step in the paper is to generalise the *ω-functions*; and all the solutions of the equation $\nabla^2 u = \partial u/\partial t$ are found which are of the form $f(t)\,\phi\,(r^2/t)\,\Theta$, where Θ is a function of the angular co-ordinates of the point

alone. The result is that, as in other problems, Θ is a spherical harmonic, or circular function, in the case of three and two dimensions respectively, that $f(t)$ is of the form t^{-n}, while $\phi(r^2/4t)$ satisfies the generalised equation,

$$d^2y/dx^2+(z+z^{-1})\,dy/dz+(2n-m^2z^{-2})\,y = 0,$$

where $z = rt^{-\frac{1}{2}}$ or $xd^2y/dx^2+(x+1+m)\,dy/dx+(n+\frac{1}{2}m)\,y = 0$, if $x = r^2/4t$. This equation is solved by the Laplace transformation, and the solution, $(2\pi\iota)^{-1}e^{-x}\int e^{-ux}(u+1)^{n+\frac{1}{2}m-1}u^{\frac{1}{2}m-n}du$, taken round a loop contour from infinity round $u = 0$, is denoted by $\omega_{n-1,m}$. The asymptotic expansion of $\omega_{n,m}$ is

$$\{e^{-x}(-x)^{n-\frac{1}{2}m}/\Gamma(n-\frac{1}{2}m+1)\}$$
$$\left[1-\frac{n^2-(\frac{1}{2}m)^2}{x}+\frac{\{n^2-(\frac{1}{2}m)^2\}\{(n-1)^2-(\frac{1}{2}m)^2\}}{x^2}-\ldots\right],$$

which terminates if either $n-\frac{1}{2}m$ or $n+\frac{1}{2}m$ is a positive integer.

Thus the solutions of $\nabla^2u = \partial u/\partial t$ in the cases of one, two, and three dimensions respectively are

$$t^{-(n+1)}x^{\frac{1}{2}}\omega_{n+\frac{1}{4},\frac{1}{2}}(x),\quad t^{-(n+1)}x^{\frac{1}{2}m}\omega_{n,m}(x)\begin{matrix}\cos\\ \sin\end{matrix}m\theta,\quad t^{-(n+1)}x^{\frac{1}{2}k}\omega_{n-\frac{1}{4},k+\frac{1}{2}}(x)\,\gamma_k(\theta,\phi),$$

γ_k being a spherical harmonic of order k, which reduce for certain values of the constants to simple well-known solutions, viz.:—

(i) To $(-\pi t)^{-\frac{1}{2}}e^{-x}$ (for $n = -\frac{1}{2}$), (ii) to $t^{-1}e^{-x}$ (for $n = m = 0$), (iii) to $t^{-\frac{3}{2}}e^{-x}$ (for $n = \frac{1}{2}$, $k = 0$).

The one-dimensional solution obtained is not new, it practically coincides with the function of the parabolic cylinder, or the functions considered by Sturm* in connection with the conduction of heat, by Thiele and Charlier in reference to probability, and by Hermite.† The two-dimensional functions seem to be new, but they are shown to be the equivalent in polar co-ordinates of the functions considered by Hermite

$$(\partial'/\partial'x)^m(\partial/\partial y)^n\,e^{-\frac{1}{2}(x^2+y^2)},$$

and the relations connecting the two types of function are obtained.

It is shown in the paper that any function of x and θ can be expanded in the form

$$\Sigma\Sigma a_{n,m}\omega_{n,m}\cos m\theta,\quad\text{where}\quad \overset{m}{(-)}\,\pi a_{n,m} = \int_0^{2\pi}\int_0^{\omega} fe^x x^{-\frac{1}{2}m}\omega_{n,-m}\cos m\theta\,dx\,d\theta,$$

and a proof of the convergence and validity of the expansion is given in the case where f is independent of θ, with an indication of the extension to the general case. Hence a series is obtained giving a solution of the equation

* Liouville, I and II.

† 'Comptes Rendus,' vol. 58.

$\nabla^2 u = \partial u/\partial t$, which reduces for $t = t_0$ to a function given all over the plane, viz. :—

$\Sigma\Sigma a_{n,m}(t_0/t)^{n+1}\,\omega_{n,m}(r^2/4t_0)\cos m\theta$, the coefficient $a_{n,m}$ being obtained as above. This series is shown, by direct expansion, to be equivalent to Laplace's definite integral solution of the cooling of an infinite plate, viz.,

$$\iint f(\xi,\eta)\,e^{-[(x-\xi)^2+(y-\eta)^2]/4(t-t_0)}\,d\xi d\eta/4\pi(t-t_0).$$

Various other expansions are given analogous to Neumann's expansions of $J_n(r^2+\rho^2-2r\rho\cos\theta)^{\frac{1}{2}}$ in a series of Bessel functions.

The last part of the paper considers the equivalent functions in Cartesian co-ordinates, in terms of which Hermite has expanded any function. It is shown that an equivalent method of obtaining Hermite's expansion is that of equating the successive moments of the series to those of the given function, with special reference to obtaining the best fitting surface for a given frequency distribution in two dimensions. The series of approximations so obtained is compared with that of Edgeworth,* which is ultimately the same, and is shown to converge rapidly under the same conditions. An example is then given of the contour lines of a surface obtained by taking terms up to the fourth order. These contour lines show much similarity with those actually observed in statistical work, such as those given by Perozzo in his memoir.† The possibility of obtaining a fairly close approximation to a given distribution by means of a knowledge of the product moments up to the fourth order is thus established.

II.—*The Derivation of the Generalised ω-function.*

1. *The Fundamental Differential Equation.*—In order to obtain the types of function suitable to the problems mentioned, the most general solution of the equation $\nabla^2 u = \partial u/\partial t$ will be found in the form $T.R.\Theta$, where T is a function of t alone, R a function of r^2/t, and Θ of the angular co-ordinates. To begin with, the cases of one, two, and three dimensions are considered separately.

(*a*) *One Dimension.*—Putting $z = r^2/t$ in the equation, it becomes

$$4z\frac{\partial^2 u}{\partial z^2}+(z+2)\frac{\partial u}{\partial z} = t\,\frac{\partial u}{\partial t}.$$

If $u = Z.T$, Z and T being functions of z and t respectively, we have

$$\left\{4z\frac{\partial^2 Z}{\partial z^2}+(z+2)\frac{\partial Z}{\partial z}\right\}Z^{-1} = \frac{t}{T}\frac{\partial T}{\partial t}, \tag{1}$$

so that both sides must be independent of t and z.

* 'Camb. Phil. Trans.,' August, 1904.

† 'Annali di Statistica,' 3A, 5, 1883.

Hence $\quad t\dfrac{\partial \mathrm{T}}{\partial t}+n\mathrm{T} = 0 \quad$ and $\quad 4z\dfrac{\partial^2 \mathrm{Z}}{\partial z^2}+(z+2)\dfrac{\partial \mathrm{Z}}{\partial z}+n\mathrm{Z} = 0,$

where n is any constant. Thus $\mathrm{T} = t^{-n}$.

(*b*) *Two Dimensions.*—The equation in polar co-ordinates is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial t}.$$

If $u = \Theta v$ satisfies this equation, Θ being a function of θ only, and v being independent of θ, the functions Θ and v satisfy

$$\frac{r^2}{v}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right) = -\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = m^2,$$

m being a constant. Thus $\Theta = \mathrm{A}\sin m\theta+\mathrm{B}\cos m\theta$, and again, putting $z = r^2/t$, and $v = \mathrm{T}\,.\,\mathrm{Z}$, the same procedure as above gives $\mathrm{T} = t^{-n}$ and

$$4z\frac{\partial^2 \mathrm{Z}}{\partial z^2}+(z+4)\frac{\partial \mathrm{Z}}{\partial z}+\left(n-\frac{m^2}{z}\right)\mathrm{Z} = 0. \tag{2}$$

(*c*) *Three Dimensions.*—Exactly similar procedure in this case leads to the conclusion that, if $u = \mathrm{T}\,.\,\mathrm{Z}\,.\,\Theta$, Θ being a function of θ and ϕ, Θ must be a spherical harmonic—say of order k, T must be t^{-n}, and that then Z satisfies the equation

$$4z\frac{\partial^2 \mathrm{Z}}{\partial z^2}+(z+6)\frac{\partial \mathrm{Z}}{\partial z}+\left\{n-\frac{k\,(k+1)}{z}\right\}\mathrm{Z} = 0. \tag{3}$$

(*d*) *Three Dimensions—Cylindrical Co-ordinates* $(\rho,\,\theta,\,\zeta)$.—In this system of co-ordinates the general solution is $\begin{matrix}\sin\\ \cos\end{matrix} m\theta \begin{matrix}\sinh\\ \cosh\end{matrix} k\zeta\mathrm{Z}$, where Z satisfies the equation

$$4z\frac{\partial^2 \mathrm{Z}}{\partial z^2}+(z+4)\frac{\partial \mathrm{Z}}{\partial z}+\left(k^2+n-\frac{m^2}{z}\right)\mathrm{Z} = 0. \tag{4}$$

The equations (1) to (4) can be brought under the same form. Putting $\mathrm{Z} = z^{\frac{1}{4}}\mathrm{X}$, $z^{\frac{1}{2}m}\mathrm{X}$, $z^{\frac{1}{2}k}\mathrm{X}$, $z^{\frac{1}{2}m}\mathrm{X}$ in the four cases respectively, and also $z = 4x$, the equations become, using dashes to denote differential coefficients,

$$x\mathrm{X}''+(x+\tfrac{3}{2})\,\mathrm{X}'+(n+\tfrac{1}{2})\,\mathrm{X} = 0, \tag{5}$$

$$x\mathrm{X}''+(x+1+m)\,\mathrm{X}'+(n+\tfrac{1}{2}m)\,\mathrm{X} = 0, \tag{6}$$

$$x\mathrm{X}''+(x+\tfrac{3}{2}+k)\,\mathrm{X}'+(n+\tfrac{1}{2}k)\,\mathrm{X} = 0, \tag{7}$$

$$x\mathrm{X}''+(x+1+m)\,\mathrm{X}'+(n+k^2+\tfrac{1}{2}m)\,\mathrm{X} = 0. \tag{8}$$

Taking (6) as the standard form, the particular solution dealt with in this paper will be denoted by $\boldsymbol{\omega}_{n-1,\,m}$: the corresponding solutions of (5), (7), and (8) are therefore

$$\boldsymbol{\omega}_{n-\frac{3}{4},\,\frac{1}{2}}, \qquad \boldsymbol{\omega}_{n-\frac{5}{4},\,\frac{1}{2}k}, \qquad \boldsymbol{\omega}_{k^2+n-1,\,m},$$

and the complete solutions of the original equations are

$$(9)\ t^{-(n+1)}x^{\frac{1}{2}}\omega_{n+\frac{1}{4},\frac{1}{2}}(x), \qquad (10)\ t^{-(n+1)}x^{\frac{1}{2}m}\begin{matrix}\sin\\ \cos\end{matrix}\, m\theta\ \omega_{n,m}(x),$$

$$(11)\ t^{-(n+1)}x^{\frac{1}{2}k}\omega_{n-\frac{1}{4},k+\frac{1}{2}}(x)\,y_k(\theta), \qquad (12)\ t^{-(n+1)}x^{\frac{1}{2}m}\omega_{n+k^2,m}(x)\begin{matrix}\cosh\\ \sinh\end{matrix}\,kz\begin{matrix}\cos\\ \sin\end{matrix}\,m\theta.$$

The simplest solutions are given in the four cases as follows:—

(i) $n = -\frac{1}{2}$, (ii) $n = 0, m = 0$, (iii) $n = \frac{1}{2}, k = 0$, (iv) $n = m = k = 0$.

2. *The Solution of the Fundamental Equation.*—Putting $X = \int_c e^{-vx}V dv$ and forming the Laplace transformation of the equation, we find that V must satisfy the equation

$$v(v-1)\,\partial V/\partial v = \{(m-1)\,v-(n+\tfrac{1}{2}m-1)\}\,V,$$

giving

$$V = Av^{n+\frac{1}{2}m-1}(v-1)^{\frac{1}{2}m-n},$$

while the contour c must be such that $Ve^{-vx}(v-v^2)$ has the same value at the two extremities. Assuming x to be real and positive, this will be attained by taking a contour consisting of a line following the real axis from $v = \infty$ to $v = a\,(a<1)$, then a circle about $v = 1$ of radius a, and a straight line returning along the real axis to infinity. A similar contour encircling the point 0 instead of 1 will be denoted by γ. Putting $u = v-1$ and $A = \frac{1}{2}\pi i$, we have for $\omega_{n,m}$ the expression $(\frac{1}{2}\pi\iota)e^{-x}\int_{\gamma_0} e^{-ux}(u+1)^{n+\frac{1}{2}m}u^{\frac{1}{2}m-n-1}du.$

The integral gives the asymptotic expansion for $\omega_{n,m}$

$$(13)\quad \frac{e^{-x}(-x)^{n-\frac{1}{2}m}}{\Gamma(n-\frac{1}{2}m+1)}\left\{1-\frac{(n+\frac{1}{2}m)(n-\frac{1}{2}m)}{x} + \frac{(n+\frac{1}{2}m)(n+\frac{1}{2}m-1)(n-\frac{1}{2}m)(n-\frac{1}{2}m-1)}{2!\,.\,x^2}-\dots\right\},$$

which terminates if either $n-\frac{1}{2}m$ or $n+\frac{1}{2}m$ is a positive integer

The most important case is that in which $n-\frac{1}{2}m$ is a positive integer; in that case

$$(14)\quad \omega_{n,m} = e^{-x}\,\Gamma(n+\tfrac{1}{2}m+1)\left\{\frac{1}{\Gamma(m+1)(n-\frac{1}{2}m)!}-\frac{x}{\Gamma(m+2)1!(n-\frac{1}{2}m-1)!} + \dots(-)^{n-\frac{1}{2}m}\frac{x^{n-\frac{1}{2}m}}{\Gamma(n+\frac{1}{2}m+1)(n-\frac{1}{2}m)!}\right\}.$$

In particular, if $m = 0$ this reduces to Pearson's function ω_{2n}.

When $n-\frac{1}{2}m$ is half an odd integer, $\omega_{n,m}$ becomes purely imaginary. This case occurs below, but no difficulty arises on that account.

Especially important is the case in which both $n-\frac{1}{2}m$ and $n+\frac{1}{2}m$ are integers, for then we have the relation,

$$(15)\quad (-x)^{\frac{1}{2}m}\omega_{n,m}/\Gamma(n+\tfrac{1}{2}m+1) = (-x)^{-\frac{1}{2}m}\omega_{n,-m}/\Gamma(n-\tfrac{1}{2}m+1).$$

3. *Relations connecting Successive Functions.*—

(*a*) *Difference equations*—

$$2\pi\iota\, x\, d\omega_{n-1,m}/dx = -\int_\gamma (u+1)^{n-\frac{1}{2}m} u^{-n-\frac{1}{2}m} e^{-(1-u)x} x\, du$$

$$= -\int_\gamma \partial/\partial u \left\{(u+1)^{n-\frac{1}{2}m} u^{-n-\frac{1}{2}m}\right\} e^{-(1+u)x}\, du$$

$$= 2\pi\iota \left\{(n+\tfrac{1}{2}m)\,\omega_{n,m} - (n-\tfrac{1}{2}m)\,\omega_{n-1,m}\right\}.$$

Thus $$x\, d\omega_{n-1,m}/dx + (n-\tfrac{1}{2}m)\,\omega_{n-1,m} = (n+\tfrac{1}{2}m)\,\omega_{n,m}. \qquad (16)$$

Applying this twice and simplifying by means of the fundamental differential equation, we find

$$(n+\tfrac{1}{2}m)\,\omega_{n,m} - (2n-x-1)\,\omega_{n-1,m} + (n-\tfrac{1}{2}m-1)\,\omega_{n-2,m} = 0. \qquad (17)$$

(*b*) *The Expression of* $\omega_{n,m}$ *as a Differential Coefficient.*—Since the derivative of any solution of $\nabla^2(u) = \partial u/\partial t$ with respect to t is also a solution, and since $\partial/\partial t\,\{t^n f(r^2/t)\} = t^{n-1}\phi(r^2/t)$, we deduce at once, after adjusting the numerical coefficients, the following relations in the cases of one, two, and three dimensions respectively (*vide* (9), (10), (11))—

$$\partial/\partial t\,\{t^{-n} x^{\frac{1}{2}} \omega_{n-\frac{3}{4},\frac{1}{2}}(x)\} = -nt^{-n-1} x^{\frac{1}{2}} \omega_{n+\frac{1}{4},\frac{1}{2}}(x), \qquad (18)$$

$$\partial/\partial t\,\{t^{-n} x^{\frac{1}{2}m} \omega_{n-1,m}(x)\} = -(n-\tfrac{1}{2}m)\, t^{-n-1} x^{\frac{1}{2}m} \omega_{n,m}(x), \qquad (19)$$

$$\partial/\partial t\,\{t^{-n} x^{\frac{1}{2}k} \omega_{n-\frac{5}{4},k+\frac{1}{2}}(x)\} = -(n-\tfrac{1}{2}k)\, t^{-n-1} x^{\frac{1}{2}k} \omega_{n-\frac{1}{4},k+\frac{1}{2}}(x). \qquad (20)$$

III.—*The Linear Function* $\omega_{n+\frac{1}{4},\frac{1}{2}}$.

1. Restricting the work in this section to the functions arising in the one-dimensional case, which are briefly called the *linear* ω*-functions*, it will be convenient to write ω_ν for $\omega_{n-\frac{1}{4},\frac{1}{2}}$ and ω'_ν for $\omega_{n-\frac{1}{4},-\frac{1}{2}}$, and ν in place of n.

From (18) we then have

$$t^{-\nu-\frac{1}{2}} x^{\frac{1}{2}} \omega_\nu = (-\partial/\partial t)^\nu \{t^{-\frac{1}{2}} x^{\frac{1}{2}} \omega_0(x)\}\, \Gamma(\tfrac{1}{2})/\Gamma(\nu+\tfrac{1}{2})$$

$$= (-)^{\nu-\frac{1}{2}} (\partial/\partial t)^\nu (t^{-\frac{1}{2}} e^{-x})/\Gamma(\nu+\tfrac{1}{2}) = (-)^{\nu-\frac{1}{2}} (\partial/\partial r)^{2\nu} (t^{-\frac{1}{2}} e^{-r^2/4t}),$$

assuming that ν is an integer. If $\nu-\frac{1}{2}$ is an integer, we have, similarly,

$$t^{-\nu-\frac{1}{2}} x^{\frac{1}{2}} \omega_\nu = (-)^{\nu-\frac{1}{2}} (\partial/\partial t)^{\nu-\frac{1}{2}} \{t^{-1} x^{-\frac{1}{2}} \omega_{\frac{1}{2}}(x)\}/\Gamma(\nu+\tfrac{1}{2})$$

$$= (-)^{\nu-\frac{1}{2}} (\partial/\partial r)^{2\nu-1} \{\tfrac{1}{2} t^{-\frac{3}{2}} r e^{-r^2/4t}\} = (-)^{\nu-\frac{1}{2}} (\partial/\partial r)^{2\nu} (t^{-\frac{1}{2}} e^{-r^2/4t}).$$

Thus in either case

$$\omega_{\frac{1}{2}\nu} = (-)^{\nu-\frac{1}{2}} t^{\frac{1}{2}\nu} (\partial/\partial r)^\nu (e^{-r^2/4t})/r\Gamma\{\tfrac{1}{2}(\nu+1)\}. \qquad (21)$$

Thus, as a function of r, $\omega_{\frac{1}{2}r}$ is identified with the function considered by Hermite.*

* 'Comptes Rendus,' vol. 68.

2. *An Expansion in a Series of Linear ω-Functions.*—The expansion of $\omega_{\frac{1}{2}\nu}$ is $\dfrac{e^{-x}(-x)^{\frac{1}{2}(\nu-1)}}{\Gamma\{\frac{1}{2}(\nu+1)\}}\left\{1-\dfrac{\nu(\nu-1)}{4x}+\dfrac{\nu(\nu-1)(\nu-2)(\nu-3)}{2!(4x)^2}\ldots\right\}$, which, if ν is an odd integer, becomes

$$e^{-x}\frac{\nu!}{2^{\nu-1}\Gamma\{\frac{1}{2}(\nu+1)\}}\left\{\frac{1}{\frac{1}{2}(\nu-1)!}-\frac{4x}{\frac{1}{2}(\nu-3)!\,3!}+\ldots\frac{(-4x)^{\frac{1}{2}(\nu-1)}}{\nu!}\right\},$$

and if ν is an even integer,

$$e^{-x}\frac{\nu!(-x)^{-\frac{1}{2}}}{2^{\nu}\Gamma\{\frac{1}{2}(\nu+1)\}}\left\{\frac{1}{\frac{1}{2}\nu!}-\frac{4x}{\frac{1}{2}(\nu-2)!\,2!}+\ldots-\frac{(4x)^{\frac{1}{2}\nu}}{\nu!}\right\}.$$

Hence, $\omega_{\frac{1}{2}\nu}=$ coefficient of q^ν in $\nu!\,e^{q^2+2\iota q\sqrt{x}-x}/(-x)^{\frac{1}{2}}2^\nu\Gamma\{\frac{1}{2}(\nu+1)\}$,

and therefore

$$x^{-\frac{1}{2}}e^{q^2-x}\cos 2q\sqrt{x}=i\Sigma(2q)^\nu\Gamma\{\tfrac{1}{2}(\nu+1)\}\,\omega_{\frac{1}{2}\nu}/\Gamma(\nu+1)=\iota\Sigma q^\nu\omega_{\frac{1}{2}\nu}\Gamma(\tfrac{1}{2})/\Gamma(\tfrac{1}{2}\nu+1) \tag{22}$$

and

$$x^{-\frac{1}{2}}e^{q^2-x}\sin 2q\sqrt{x}=\Sigma(2q)^\nu\Gamma\{\tfrac{1}{2}(\nu+1)\}\,\omega_{\frac{1}{2}\nu}/\Gamma(\nu+1)=\Sigma q^\nu\omega_{\frac{1}{2}\nu}\Gamma(\tfrac{1}{2})/\Gamma(\tfrac{1}{2}\nu+1), \tag{23}$$

the former summation being for even values of ν and the latter for odd, or together,

$$x^{-\frac{1}{2}}e^{(q+\iota x)^2}=\sqrt{-\pi}\sum_{\nu=0}^{\infty}(q)^\nu\omega_{\frac{1}{2}\nu}/\Gamma(\tfrac{1}{2}\nu+1). \tag{24}$$

3. *Application of the above Expansion.*—The foregoing properties will now be applied to obtain an expansion of Laplace's solution of the problem of the cooling of an infinite bar. We have

$$\{2\pi(t-t_0)\}^{-\frac{1}{2}}e^{-(r-\rho)^2/4(t-t_0)}=(2\pi)^{-1}\int_{-\infty}^{\infty}e^{-u^2(t-t_0)-\iota u(r-\rho)}du$$

$$=(4\pi)^{-\frac{1}{2}}\,i\int_{-\infty}^{\infty}e^{-u^2t-iur}\rho e^{\rho^2/4t_0}\,\Sigma u^\nu t_0^{(\nu-\frac{1}{2})}\omega_{\frac{1}{2}\nu}(\rho^2/4t_0)\,du/\Gamma(\tfrac{1}{2}\nu+1),$$

$$=(4\pi)^{-\frac{1}{2}}\rho e^{\rho^2/4t_0}\,\Sigma(-t_0)^{\frac{1}{2}(\nu-1)}\omega_{\frac{1}{2}\nu}/\Gamma(\tfrac{1}{2}\nu+1)(\partial/\partial r)^\nu\int_{-\infty}^{\infty}e^{-u^2t-\iota ur}du,$$

$$=(2t_0)^{-\frac{1}{2}}e^{\rho^2/4t_0}(\rho^2/4t_0)^{\frac{1}{2}}(r^2/4t)^{\frac{1}{2}}\Sigma\,(t_0/t)^{(\nu+\frac{1}{2})}\,\omega_{\frac{1}{2}\nu}(\rho^2/4t_0)\,\omega_\nu(r^2/4t)\;\Gamma\{\tfrac{1}{2}(\nu+1)\}/\Gamma(\tfrac{1}{2}\nu+1),$$

$$=(-2t_0)^{-\frac{1}{2}}e^{\xi}\,\xi^{\frac{1}{2}}x^{\frac{1}{2}}\,\Sigma\,(t_0/t)^{\nu+\frac{1}{2}}\omega'_{\frac{1}{2}\nu}(\xi)\,\omega_{\frac{1}{2}\nu}(x),$$

using (15) and putting ξ for $\rho^2/4t_0$.

Now, as Laplace as shown, $V=\int_{-\infty}^{\infty}f(\xi)\,(4\pi t)^{-\frac{1}{2}}e^{-(x-\xi)^2/4t}d\xi$ is a function of x satisfying $\partial^2V/\partial x^2=\partial V/\partial t$ and reducing when $t=0$ to $f(x)$. Putting $(t-t_0)$ in place of t, and expanding the subject of integration as above, we obtain

the following series to represent a solution of the same equation reducing when $t = t_0$, to $f(x)$:

$$\Sigma a_\nu \, (t_0/t)^{\frac{1}{2}(\nu+1)} x^{\frac{1}{2}} \omega^\nu (x), \tag{25}$$

where

$$a_\nu = -\tfrac{1}{2} i \int_{-\infty}^{\infty} f(\rho) e^{\xi} \xi^{\frac{1}{2}} \omega'_\nu (\xi) \, d\xi.$$

In accordance with a remark made above, it may be noted that, when ω_ν is real, ω'_ν is imaginary, so that the expansion is wholly real. A different method of establishing the same expansion is given below.

IV.—*The General ω-Function—Integral Properties.*

Let

$$-4\pi^2 \mathrm{I}_{n, n', m} = -4\pi^2 \int_0^\infty e^x \omega_{n, m} (x) \, \omega_{n', -m} (x) \, dx$$

$$= \int_0^\infty dx \int_\gamma du \int_\gamma dv \, e^{-(u+v+1)x} (u+1)^{n+\frac{1}{2}m} u^{\frac{1}{2}m-n-1} (v+1)^{n'-\frac{1}{2}m} v^{-\frac{1}{2}m-n'-1}.$$

The asymptotic expansion of the ω-function shows that the integration in regard to x to the infinite upper limit is valid, and the order of integration may be inverted. Thus

$$-4\pi^2 \mathrm{I}_{n, n', m} = \int_\gamma du \int_\gamma dv \, (u+1)^{n+\frac{1}{2}m} u^{\frac{1}{2}m-n-1} (v+1)^{n'-\frac{1}{2}m} v^{-\frac{1}{2}m-n'-1} (u+v+1)^{-1}.$$

For every value of v the subject of integration is zero to the second order at least when u is infinite if m is positive, and *vice versâ* if m is negative. Taking m positive for the sake of argument,

$$-4\pi^2 \mathrm{I}_{n, n', m}$$

$$= -\int_{-v-1} du \int_\gamma dv \, (u+1)^{n+\frac{1}{2}m} u^{\frac{1}{2}m-n-1} (v+1)^{n'-\frac{1}{2}m} v^{-\frac{1}{2}m-n'-1} (u+v+1)^{-1}.$$

$$= (-)^m 2\pi\iota \int_\gamma dv \, v^{n-n'-1} (1+v)^{n'-n-1} = (-)^m (-4\pi^2),$$

if $n = n'$, or $= 0$ if $n \pm n'$. Thus

$$\int_0^\infty e^x \omega_{n, m} (x) \, \omega_{n', -m} (x) \, dx = 0, \tag{26}$$

but

$$\int_0^\infty e^x \omega_{n, m} (x) \, \omega_{n, -m} (x) \, dx = (-1)^m. \tag{27}$$

Again, consider the moment integral

$$2\pi i \mathrm{M}_r = 2\pi i \int_0^\infty \omega_{n, m} (x) \, x^r dx = \int_0^\infty x^r dx \int_\gamma e^{-(u+1)x} (u+1)^{n+\frac{1}{2}m} u^{\frac{1}{2}m-n-1} du$$

$$= \Gamma(r+1) \int_\gamma (u+1)^{n+\frac{1}{2}m-r-1} u^{\frac{1}{2}m-n-1} \, du.$$

Thus, if $n+\frac{1}{2}m-r$ is a positive integer, $(1+u)^{n+\frac{1}{2}m-r-1}$ is a polynomial and the subject of integration has no residue at $u=0$. But if $n+\frac{1}{2}m-r$ is zero or a negative integer, the residue is $(-)^{n-\frac{1}{2}m}(r+1-n-\frac{1}{2}m)\ldots(r-m)/(n-\frac{1}{2}m)!$. Thus

$$M_r = 0,\ (-1)^{n-\frac{1}{2}m},\ (-)^{n-\frac{1}{2}m}\,(-)^{n-\frac{1}{2}m}\,\Gamma(r+1)\,\Gamma(r-m+1)/(n-\frac{1}{2}m)!\ \Gamma(r-n-\tfrac{1}{2}m+1), \quad (28)$$

according as $r <, =,$ or $> n+\frac{1}{2}m$. In this r is not necessarily an integer, but $n-\frac{1}{2}m$ and $n+\frac{1}{2}m-r$ are assumed to be integers.

If these results, (26) and (27), be applied in the one-dimensional case considered above, assuming that $f(\xi) = \Sigma a_\nu \xi^{\frac{1}{2}}\omega_\nu(\xi)$, we obtain at once $a_\nu = -\frac{1}{2}i\int_0^\infty f(\xi)\,e^{\xi}\xi^{\frac{1}{2}}\omega'_\nu(\xi)\,d\xi$. Consequently, if $\xi = r^2/4t_0$ and $x = r^2/4t$, $\Sigma a_\nu(t_0/t)^{n+\frac{1}{2}}x^{\frac{1}{2}}\omega_\nu(x)$ is a function satisfying $\partial^2 u/\partial r^2 = \partial u/\partial t$, and reducing when $t=t_0$ to the given function $f(x)$. Thus the expansion (25) is obtained.

V.—*The ω-Functions in a Plane.*

Coming to the special values of n and m suitable to the development of functions in a plane, m is an integer and $n-\frac{1}{2}m$ is an integer. Using relation (19), we have

$$\begin{aligned}\Omega_{n,m} &= t^{-(n+1)}(r^2/t)^{-\frac{1}{2}m}\omega_{n,m}(r^2/4t)\,e^{\iota m\theta}\\ &= (-\partial/\partial t)^n\,\{t^{-1}(r^2/t)^{-\frac{1}{2}m}\omega_{0,m}e^{\iota m\theta}\}\,\Gamma(\tfrac{1}{2}m+1)/\Gamma(\tfrac{1}{2}m+n+1)\\ &= \{-\nabla^2\}^n\{t^{-1}(r^2/t)^{-\frac{1}{2}m}\omega_{0,m}e^{\iota m\theta}\}\,\Gamma(\tfrac{1}{2}m+1)/\Gamma(\tfrac{1}{2}m+n+1). \qquad (29)\end{aligned}$$

In particular, if we start from the function independent of θ, *i.e.* put $m=0$, $\Omega_{n,0} = t^{-n+1}\omega_n(r^2/4t) = (-\partial/\partial t)^n\,(t^{-1}e^{-r^2/4t})/n!$, so that

$$\omega_n = (-t)^n/n!\,(\partial^2/\partial r^2+1/r\,\partial/\partial r)^n e^{-r^2/4t}. \qquad (30)$$

This particular set of functions may conveniently be called the *zonal* ω-functions.

If m be not zero, but $n-\frac{1}{2}m$ a positive integer, so that $\omega^{n,m}$ is polynomial,

$$\begin{aligned}\Omega_{n,m} &= (-\partial/\partial t)^{n-\frac{1}{2}m}\,\{t^{-(\frac{1}{2}m+1)}(r^2/t)^{-\frac{1}{2}m}\omega_{\frac{1}{2}m,m}e^{\iota m\theta}\}\\ &= (-\partial/\partial t)^{n-\frac{1}{2}m}\,\{r^m t^{-m-1-r^2/4t}e^{\iota m\theta}\}\,\Gamma(m+1)/\Gamma(\tfrac{1}{2}m+n+1)\\ &= (-)^{n-\frac{1}{2}m}(\nabla^2)^{n-\frac{1}{2}m}\,\{r^m t^{-m-1}e^{-r^2/4t}e^{\iota m\theta}\}\,\Gamma(m+1)/\Gamma(\tfrac{1}{2}m+n+1). \quad (31)\end{aligned}$$

Thus all the functions found are expressed in terms of the derivatives of the fundamental function with respect to r, θ, and t.

Since the derivatives of any solution of $\nabla^2 u = \partial u/\partial t$ with respect to Cartesian co-ordinates x and y are also solutions, we may expect all solutions to be expressible in terms of the derivatives with respect to x and y of the fundamental solution.

The following equations may be shown to hold for the planar ω-functions,

$$\Omega\tfrac{1}{2}m, m = (\partial/\partial x+i\partial/\partial y)^m (t^{-1}e^{-r^2/4}t),$$

and, therefore, $n-\frac{1}{2}m$ being an integer,

$$\begin{aligned}\Omega_{n,m} &= (\partial^2/\partial x^2+\partial^2/\partial y^2)^{n-\frac{1}{2}m} (\partial/\partial x+i\partial/\partial y)^m (t^{-1}e^{-r^2/4t}) \\ &= (\partial/\partial t)^{n-\frac{1}{2}m} (\partial/\partial x+i\partial/\partial y)^m (t^{-1}e^{-r^2/4t}),\end{aligned}$$

or

$$(\partial/\partial x+i\partial/\partial y)^{n+\frac{1}{2}m} (\partial/\partial x-i\partial/\partial y)^{n-\frac{1}{2}m} (t^{-1}e^{-r^2/4t})\,; \tag{32}$$

e.g., we have

$$(\partial^2/\partial x^2-\partial^2/\partial y^2)(t^{-1}e^{-r^2/4t}) = (4t^2)^{-1}(r^2/t)\,e^{-r^2/4t}\cos 2\theta.$$

$$2\,\partial^2/\partial x\partial y\,(t^{-1}e^{-r^2/4t}) = (4t^2)^{-1}(r^2/t)\,e^{-r^2/4t}\sin 2\theta,$$

and

$$(\partial^2/\partial x^2+\partial^2/\partial y^2)(t^{-1}e^{-r^2/4t}) = -(t^2)^{-1}(1-r^2/4t)\,e^{-r^2/4t},$$

giving the values of $\partial^2/\partial x^2$, $\partial^2/\partial y^2$, $\partial^2/\partial z^2$ in terms of Ω_{10} and Ω_{12}. Thus the connection is established between the planar ω-functions and Hermite's functions $\partial^{r+s}/\partial x^r\partial y^s\,\{e^{-a(x^2+y^2)}\}$, to which reference is made below.

VI.—*The Expansion of any Function in Planar ω-Functions.*

1. Suppose that a function of r and θ may be expanded in the form

$$f(r,\theta)\sum_n\sum_m a_{nm}(x)^{\frac{1}{2}m}\,\omega_{nm}(x)\cos m\theta+\sum_n\sum_m b_{n,m}x^{\frac{1}{2}m}\omega_{n,m}(x)\sin m\theta,$$

where $x = r^2/4t$, t being a parameter. Then, using the Fourier method for all values of m, we have

$$\int_0^{2\pi} f(r,\theta)\cos m\theta\,d\theta = \pi\sum_n a_{nm}x^{\frac{1}{2}m}\omega_{n,m}$$

and

$$\int_0^{2\pi} f(r,\theta)\sin m\theta\,d\theta = \pi\sum b_{n,m}x^{\frac{1}{2}m}\omega_{n,m}.$$

In order to find the coefficients $a_{n,m}$, $b_{n,m}$, we now use the integrals developed above, and obtain

$$(-)^m\,\pi a_{n,m} = \int_0^{2\pi}\int_0^{\infty} f\,.\,e^x x^{-\frac{1}{2}m}\omega_{n,-m}\cos m\theta\,dx\,d\theta, \tag{33}$$

with a similar expression for $b_{n,m}$. Thus, in order to obtain the coefficient of $\omega_{n,m}$, we make use of the allied function $\omega_{n,-m}$, which bears to it the relation (15).

The values of m and n required to be taken do not appear, but it will be shown presently that m takes all integer values and n takes those values for which $n-\frac{1}{2}m$ is zero or a positive integer. The expansion is therefore one in a series of polynomials. The corresponding expansion in Cartesian co-ordinates is given by Hermite.*

* 'Comptes Rendus,' vol. 68.

2. *The Expansion of the Laplace Definite Integral in a Plane.*—An expansion analogous to (26) will first be obtained. x standing, as everywhere, for $r^2/4t$, let z stand for $re^{\iota\theta}/2\sqrt{t}$ and z' for $re^{-\iota\theta}/2\sqrt{t}$. Then, taking all values of n and m for which $n-\frac{1}{2}m$ and $n+\frac{1}{2}m$ are both positive integers,

$$\omega_{n,m} = e^{-x}(n+\tfrac{1}{2}m)!\left\{\frac{1}{m!\,(n-\frac{1}{2}m)!} - \frac{x}{(m+1)!\,1!\,(n-\frac{1}{2}m-1)!}\cdots + \frac{(-x)^{n-\frac{1}{2}m}}{(n+\frac{1}{2}m)!\,(n-\frac{1}{2}m)!}\right\}$$

$$= (n+\tfrac{1}{2}m)!\,x^{-\frac{1}{2}m}\,.\,\text{coefficient of } h^{n-\frac{1}{2}m}k^{n+\frac{1}{2}m} \text{ in } e^{hk+(k-h)\sqrt{x}-x};$$

or
$$\sum_{n=0}^{\infty}\sum_{m=-2n}^{m=+2n} x^{\frac{1}{2}m}\omega_{n,m}h^{n-\frac{1}{2}m}k^{n+\frac{1}{2}m}/(n+\tfrac{1}{2}m)! = e^{(h+\sqrt{x})(k-\sqrt{x})}, \tag{34}$$

n taking all half-integer values from 0 to infinity.

Again, taking the same range of values for m and n,

$$\Sigma\Sigma x^{\frac{1}{2}m}e^{m\iota\theta}\omega_{nm}h^{n-\frac{1}{2}m}k^{n+\frac{1}{2}m}/(n+\tfrac{1}{2}m)! = e^{(he^{-\iota\theta}+\sqrt{x})(ke^{\iota\theta}+\sqrt{x})} = e^{(h+z)(k-z')}. \tag{35}$$

Expanding the last expression by Taylor's theorem, we have

$$x^{\frac{1}{2}m}e^{m\iota\theta}\omega_{nm} = (-)^{n+\frac{1}{2}m}(\partial/\partial z)^{n-\frac{1}{2}m}(\partial/\partial z')^{n+\frac{1}{2}m}(e^{-zz'})/(n-\tfrac{1}{2}m)!.$$

Thus, putting $\xi = re^{\iota\theta}$ and $\eta = re^{\iota\theta}$,

$$x^{\frac{1}{2}m}e^{m\iota\theta}\omega_{nm} = (-)^{n+\frac{1}{2}m}4^n t^{n+1}(\partial/\partial\xi)^{n-\frac{1}{2}m}(\partial/\partial n)^{n+\frac{1}{2}m}(t^{-1}e^{-r^2/4t})/(n-\tfrac{1}{2}m)!$$

This gives the proof of the relations given above (32).

Putting $\iota h = p+\iota q$ and $-\iota k = p-\iota q$ in (35), that expression becomes $e^{p^2+q^2+\iota(pu+qv)-x}$, where (uv) are Cartesian co-ordinates.

Hence

$$(t-t_0)^{-1}e^{-[(u-u_1)^2+(v-v_1)^2]/4(t-t_0)} = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,e^{-(p^2+q^2)(t-t_0)-\iota p(u-u_1)-\iota q(v-v_1)}$$

$$= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,e^{-(p^2+q^2)t-\iota(pu+qv)}\,e^{x_1}\Sigma\Sigma x_1^{\frac{1}{2}m}e^{m\iota\phi}\omega_{nm}(x_1)\,h^{n-\frac{1}{2}m}k^{n+\frac{1}{2}m}/(n+\tfrac{1}{2}m)!$$

by (35), where $x_1 = (u_1^2+v_1^2)/4t_0$. Now put $h' = h\sqrt{(t/t_0)}$ and $k' = k\sqrt{(t-t_0)}$ and $h' = p+\iota q$, $k' = p-\iota q$. Then the above becomes

$$\Sigma\Sigma(-)^{n-\frac{1}{2}m}(\partial/\partial z)^{n-\frac{1}{2}m}(\partial/\partial z')^{n+\frac{1}{2}m}(2\pi)^{-1}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(p^2+q^2)t-\iota(px+qy)}dp\,dq\,t_0^{\,n}t^{-n-1}e^{x_1}x_1^{\frac{1}{2}m}e^{m\iota\phi}\omega_{nm}(x_1)/(n+\tfrac{1}{2}m)!$$

$$= \Sigma\Sigma(-)^{-2n}e^{x_1}x^{-\frac{1}{2}m}x_1^{\frac{1}{2}m}e^{m\iota(\phi-\theta)}\omega_{n,-m}(x)\,\omega_{n,m}(x_1)\,t_0^{\,n}t^{-n-1}.$$

The summation throughout is for all values of m and n such that $n+\frac{1}{2}m$ and $n-\frac{1}{2}m$ are positive integers. There will thus correspond to each term in the summation another with an equal but opposite value of m, and, using (17)

these terms will only differ by $e^{m\iota(\phi-\theta)}$ being changed to $e^{-m\iota(\phi-\theta)}$. Remembering also that $n-\frac{1}{2}m$ is an integer, $(-)^{-2n}=(-)^{-m}$. Hence

$$(t-t_0)^{-1}e^{-[(u-u_1)^2+(v-v_1)^2]/4(t-t_0)}$$

$$=\sum_{n=0}^{\infty} e^{x_1}\omega_{n,0}(x)\,\omega_{n.0}x_1 t_0{}^n t^{-n-1}$$
$$+2\Sigma\Sigma(-)^m e^{x_1}x^{\frac{1}{2}m}x_1{}^{-\frac{1}{2}m}\omega_{nm}(x)\,\omega_{n,-m}(x_1)\cos m(\phi-\theta)\,t_0{}^n t^{-h-1}. \quad (36)$$

This expansion is suggestive of Neumann's expansion in the theory of the Bessel function. Another theorem is given below, which, in some respects, is a nearer analogue. The development of the Laplace definite integral follows at once from (36) and is identical with that in the last paragraph (33). The method of this paragraph shows that the values of m and n are as there stated.

3. *The Expansion of* $\omega_{m,n}\{(r^2-2r\rho\cos(\phi-\theta)+\rho^2)\}/4t$ *in a Series of* ω*-Functions.*—As in the last section,

$$t^{-1}e^{-[(x-x_1)^2+(y-y_1)^2]\,4t}$$
$$=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,e^{-(p^2+q^2)t-\iota p(u-u_1)-\iota q(v-v_1)}$$
$$=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,e^{-(p^2+q^2)t-\iota pu-\iota qv}$$
$$\Sigma\Sigma(-h)^{n-\frac{1}{2}m}k^{n+\frac{1}{2}m}\eta'^{\,n-\frac{1}{2}m}\xi^{n+\frac{1}{2}m}/(n-\tfrac{1}{2}m)!\,(n+\tfrac{1}{2}m)!\,4^n$$

(the summation being taken for the same values of m and n as above)

$$=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,(\partial/\partial\xi)^{n-\frac{1}{2}m}(\partial/\partial\eta)^{n+\frac{1}{2}m}e^{-(p^2+q^2)t-\iota(pu+qv)}$$
$$\rho^{2n}e^{m\iota\phi}/(n-\tfrac{1}{2}m)!\,(n+\tfrac{1}{2}m)!$$
$$=\Sigma\Sigma(-)^{n+\frac{1}{2}m}4^{-n}t^{-n-1}x^{\frac{1}{2}m}\omega_{n,m}e^{m\iota(\phi-\theta)}\rho^{2n}/(n-\tfrac{1}{2}m)!$$
$$=\Sigma\Sigma(-)^{n+\frac{1}{2}m}t^{-1}(\rho^2/4t)^{-n}x^{-\frac{1}{2}m}\omega_{n,m}(x)\cos m(\phi-\theta)/(n+\tfrac{1}{2}m)!,$$

where the summation extends only to values of n such that $n-\frac{1}{2}m$ is a positive integer. Thus $\omega_{0;0}\{(r^2-2r\rho\cos(\phi-\theta)+\rho^2)/4t\}$ is expanded in a series of ω-functions in the variable $(r^2/4t)$, and from this may be derived, by means of relations (32), the expansions of $\omega_{n,m}\{(r^2-2r\rho\cos(\phi-\theta)+\rho^2)/4t\}$, using differentiations with regard to t, x_1, and y_1.

4. *The Convergence and Validity of the Expansion of a Function in a Series of* ω*-functions.*—The proof of the validity of expansions in a series of normal functions is always a difficult matter, and a general theorem to cover them has yet to be established. A discussion is here given of the expansion in a series of zonal ω-functions of a function symmetrical round a point, which seems capable of extension to the general ω-functions in a plane without much increase of difficulty, though with considerable addition to the analysis. The multiplication of different discussions for each particular set of normal

functions seems undesirable, however, and the present investigation will therefore be limited to the single case of the zonal functions. A different method appears possible in the discussion of the validity of the processes in the development of the Laplace integral given above.

In the following, the basis of the work is the definite integral expression for the ω-function, viz., $(2\pi\iota)^{-1}\int_\gamma e^{-x(1+u)}(u+1)^n u^{-n-1}du.$

The expansion to be justified is

$$f(x) = \sum_0^\infty a_r \omega_r(x), \quad \text{where} \quad a_r = \int_0^\infty f(y)\,\omega_r(y)\,e^y dy.$$

Consider the sum of $(n+1)$ terms of the series

$$\sum_0^\infty a_r\omega_r(x) = \sum_0^n \int_0^\infty f(y)\,e^y\omega_r(x)\,\omega_r(y)\,dy$$

$$= -(4\pi^2)^{-1}\int_0^\infty e^y f(y)\,dy\int_\gamma du\int_\gamma dv\, e^{-(u+1)y-(v+1)x} \sum_0^n (u+1)^r(v+1)^r u^{-r-1}v^{-r-1}$$

$$= -(4\pi^2)^{-1}\int_0^\infty e^{-x}f(y)\,dy\int_\gamma du\int_\gamma dv\,(u+v+1)^{-1}$$

$$\{(u+1)^{n+1}(v+1)^{n+1}u^{-n-1}v^{-n-1}-1\}\,e^{-uy-vx} = -(4\pi^2)^{-1}\{\mathrm{I}_{n+1}-\mathrm{I}_0\}.$$

A necessary condition is at once obvious for the validity of the processes involved. For, if y is very large, since the u-contour includes points for which the real part of u is negative, e^{-uy} is not everywhere finite on that contour. But if $f(y)$ be such that, no matter how great y may be, $f(y) < e^{ky}$, where k is some finite negative constant, we may take the u-contour such that the real part of u is always greater than k, and then the integrations required are justified.

Let

$$\mathrm{J}_r(x, y) = \int_\gamma\int_\gamma e^{-uy-vx}(u+1)^r(v+1)^r u^{-r}v^{-r}(u+v+1)^{-1}du\,dv.$$

Then, if y is finite, J_0 is zero, since the u-contour includes no point at which $(u+v+1)$ vanishes for any value of v on the contour γ. Also, by the condition assumed to be satisfied by $f(y)$, the part of I_0 arising from values of y greater than a finite magnitude η can be made as small as we please by taking η sufficiently large. Thus $\mathrm{I}_0 = 0$. In treating I_{n+1} it has to be remembered that the limit of the integral is sought when n tends to infinity, and that since the contour γ encircles the point 0, and the two contours for u and v have been assumed such that $u+v+1$ does not vanish for a pair of values within these contours, $|\,(u+1)/u\,|$ and $|\,(v+1)/v\,|$ are both greater

than unity on the major portions of the contours, so that the subject of integration therefore becomes indefinitely great as n increases.

Consider now the integral $\int_0^\infty J_r(x, y)\, dy$ and divide into two portions for which y is greater and less than x respectively, calling them K_1 and K_2.

Let K_a be the same integral with the limits a and x $(a < x)$.

Then

$$\begin{aligned} K_a &= \int_a^x dy\, e^{-x} \int_\gamma du \int_\gamma dv\, e^{-vx-uy} (u+1)^r (v+1)^r u^{-r} v^{-r} (u+v+1)^{-1} \\ &= -\int_\gamma du \int_\gamma dv\, e^{-x(1+u+v)} (u+1)^r (v+1)^r u^{-r-1} v^{-r} (u+v+1)^{-1} \\ &\quad + \int_\gamma du \int_\gamma dv\, e^{-x(1+v)-au} (u+1)^r (v+1)^r u^{-r-1} v^{-r} (u+v+1)^{-1} \\ &= -{}_1K_a + {}_2K_a \text{ (say).} \end{aligned}$$

Consider the latter part ${}_2K_a$; it is absolutely and uniformly convergent, and may therefore be written

$$-\partial/\partial x \int_\gamma \int_\gamma du\, dv\, e^{-x(1+v)-au} (u+1)^r (v+1)^{r-1} u^{-r+1} v^{-r} (u+v+1)^{-1} = -\partial L_r/\partial x$$

(say). Within the contours considered there is no pair of points for which $(u+v+1) = 0$. If the u-contour be extended so as to include all the points at which $(u+v+1)$ vanishes for points on the v-contour, we add to L_r the integral

$\int_\gamma \int_{\gamma_1} e^{-x(1+v)-au} (u+1)^r u^{-r-1} (v+1)^{r-1} v^{-r} (u+v+1)^{-1}\, du\, dv$, the contour γ_1 including $u = -v-1$, but not $u = 0$, and this addition

$$= -2\pi\iota \int_\gamma e^{-(x-a)(1+v)} (1+v)^{-2} dv = 0,$$

since the contour γ does not include the point (-1), and since $x-a$ is positive. Thus the u-contour in L_r may be supposed extended so as to include the point at which $(u+v+1)$ vanishes for all values of v or γ.

Let the v-contour be now specified more precisely as that part of a circle of large radius R, with centre at $v = -\frac{1}{2}$, that lies on the positive side of the straight line on which the real part of $v = -\frac{1}{2}$, together with that part of this line which lies within that circle. This contour is considered in three parts, α, β, γ, as follows: α, the straight portion; β, that part of the curved portion adjacent to α and subtending a small angle δ at the centre of the circle; γ, the remainder.

For values of v on α, the u-contour, extended to include $-(1+v)$, is taken to be a contour just surrounding the complete v-contour. For values of v on β and γ this contour is extended in the negative direction so as always

to be a short distance on the negative side of $u = -v-1$. The u-contour is considered in three portions similar to α, β, γ, viz.: α_1, the straight portion; β_1, the arcs contiguous to α_1 and subtending an angle δ at the middle point of α_1; γ_1, the remainder. Thus the whole of the double integral is divided into nine portions obtained by combining one of the partial contours α, β, γ with one of the partial contours $\alpha_1, \beta_1, \gamma_1$.

Now $e^{-r/v}(1+v^{-1})^r$ remains always finite as r and v become large, except for $v = -1$, a value which lies outside the region under consideration. Thus $|e^{-xv}(1+v^{-1})^r| < \mathrm{A}|e^{-xv+r/v}|$ and $|e^{-au}(1+u^{-1})^r| < \mathrm{B}|e^{-au+r/u}|$. Thus, if $v = \mathrm{R}e^{\iota\theta}|e^{-xv}(1+v^{-1})^r| < \mathrm{A}e^{(-x\mathrm{R}+r/\mathrm{R})\cos\phi}$, and, if $u = \mathrm{R}'e^{\iota\phi}$; $|e^{-au}(1+u^{-1})^r| < \mathrm{B}e^{(-a\mathrm{R}'+r/\mathrm{R}')\cos\phi}$. Hence, if

$$r/\mathrm{R}'^2 = c < a < x,\ |e^{-au}(1+u^{-1})^r| < \mathrm{B}e^{-(a-c)\mathrm{R}'\cos\phi},$$

and therefore, on the arc γ, $\cos\phi$ being $> \delta$,

$$|e^{-au}(1+u^{-1})^r| < \mathrm{B}e^{-(a-c)\mathrm{R}'\delta} < \mathrm{B}e^{-(a-c)\rho\delta^{-\frac{1}{2}}} \text{ (where } \mathrm{R}' = \rho\delta^{-\frac{3}{2}}),$$

and this quantity tends to zero as δ diminishes, provided ρ remains finite.

On the arc β_1, the expression certainly remains finite.

On the straight part α_1, $|(u+1)/u|$ takes its greatest value at the point where it adjoins β_1, *i.e.* $\{(u+1)/u\}^r$ is finite on α_1, but along a length $(k\mathrm{R}')^{\frac{1}{2}}$ of α measured from the real axis it becomes comparable with $(1-1/k\mathrm{R}')^r$, *i.e.* with $(1-1/k\mathrm{R}')^{c\mathrm{R}'^2}$, *i.e.* with $e^{-c\mathrm{R}'}$, where R' is large and c finite. Similar considerations apply to the various portions of the v-contour, and putting them together we deduce that each of the portions of the double integral tends to zero as r tends to infinity. Thus $\mathrm{L}t_{r=\infty}\mathrm{L}_r = 0$ for all values of x, and therefore ${}_2\mathrm{K}_a = -\partial\mathrm{L}_r/\partial_x$ also tends to zero as r becomes infinite, for any value of a less than x, and in particular if $a = 0$.

Thus

$$\begin{aligned}\mathrm{K}_a &= \int_a^x\int_\gamma\int_{\gamma_1} e^{-x(1+v)-uy}(u+1)^r(v+1)^r u^{-r}v^{-r}(u+v+1)^{-1}du\,dv\,dy\\ &= -\int_\gamma\int_\gamma e^{-(1+u+v)x}(u+1)^r(v+1)^r u^{-r-1}u^{-r}(u+v+1)^{-1}\,du\,dv,\end{aligned}$$

and is independent of the lower limit a.

The u and v-contours being the same, this last integral will be unaltered if u and v be interchanged. Hence

$$\begin{aligned}\mathrm{K}_a &= -\tfrac{1}{2}\int_\gamma\int_{\gamma_1} e^{-x(1+u+v)}(u+1)^r(v+1)^r u^{-r}v^{-r}(u+v+1)^{-1}(u^{-1}+v^{-1})\,du\,dv\\ &= -\tfrac{1}{2}\int_\gamma\int_{\gamma_1} e^{-x(1+u+v)}(u+1)^r(v+1)^r u^{-r-1}v^{-r-1}\{1-(u+v+1)^{-1}\}\,du\,dv\\ &= \tfrac{1}{2}\{\mathrm{H}+\partial\mathrm{H}/\partial x\},\end{aligned}$$

where

$$\mathrm{H} = \int_\gamma\int_{\gamma_1} e^{-x(1+u+v)}(u+1)^r(v+1)^r u^{-r-1}v^{-r-1}(u+v+1)^{-1}\,du\,dv$$

$$= -\int_\gamma\int_{\gamma_1} e^{-x(1+u+v)}(u+1)^r(v+1)^r u^{-r-1}v^{-r-1}(u+v+1)^{-1}\,du\,dv+\epsilon,$$

where ϵ tends to zero as r becomes large; since, by reasoning similar to that used above, the double integral H with the v-contour extended to include the point $-(1+u)$ vanishes as r tends to infinity.

Thus

$$\mathrm{Lt}_{r=\infty}\mathrm{H} = -2\pi\iota\int_\gamma -u^{-1}(u+1)^{-1}\,du = -4\pi^2 \quad\text{and}\quad \mathrm{Lt}_{r=\infty}\partial\mathrm{H}/\partial x = 0.$$

Finally, therefore, $\mathrm{Lt}_{r=\infty}\mathrm{K}_a = -2\pi^2$, whatever the value of a.

Apply now the same treatment to the integral K_b, viz.:—

$$\int_x^b\int_\gamma\int_\gamma e^{-x(1+v)-uy}(u+1)^r(v+1)^r u^{-r}v^{-r}(u+v+1)\,du\,dv\,dy$$

$$= \int_\gamma\int_\gamma e^{-x(1+u+v)}(u+1)^r(v+1)^r u^{-r-1}v^{-r}(u+v+1)^{-1}\,du\,dv$$

$$-\int_\gamma\int_\gamma e^{-x(1+v)-bu}(u+1)^r(v+1)^r u^{-r-1}v^{-r}(u+v+1)^{-1}du\,dv.$$

The coefficient of u in the exponential in the second integral being now greater than that of v, we carry out first the integration in regard to u. The first part of K_b is, as above, equal to $2\pi^2$. Thus

$$\mathrm{K}_b = 2\pi^2-\int_\gamma\int_\gamma e^{-x(1+v)-bu}(u+1)^r(v+1)^r u^{-r-1}v^{-r}(u+v+1)^{-1}du\,dv.$$

In extending the v-contour of this last integral, instead of the u-contour as in K_a, we add to it

$$\int_\gamma du\int_{\gamma_1} e^{-x(1+v)-bu}(u+1)^r(v+1)^r u^{-r-1}v^{-r}(u+v+1)^{-1}\,du\,dv$$

$$= 2\pi\iota\int_\gamma du\, e^{-(b-x)u}/u = -4\pi^2;$$

γ_1, as above, standing for a contour encircling $-(1+u)$, and not zero. As before, the extended integral is zero, and therefore the unextended integral equals $4\pi^2$. Thus $\mathrm{K}_b = 2\pi^2-4\pi^2 = -2\pi^2$ for all values of b.

We have

$$\mathrm{Lt}_{r=\infty}\int_0^x f(y)\,dy\int_\gamma\int_\gamma e^{-uy-x(1+v)}(u+1)^r(v+1)^r u^{-r}v^{-r}(u+v+1)^{-1}\,du\,dv$$

$$= -2\pi f(x-)$$

and

$$\mathrm{Lt}_{r=\infty}\int_x^\infty f(y)\,dy\int_\gamma\int_\gamma e^{-uy-x(1+v)}(u+1)^r(v+1)^r u^{-r}v^{-r}(u+v+1)^{-1}\,du\,dv$$
$$= 2\pi^2 f(x+);$$

and, therefore, finally

$$\mathrm{Lt}_{n=\infty}\sum_0^n a_r\omega_r x = \tfrac{1}{2}\{f(x-)+f(x+)\},$$

and if the function f is continuous,

$$\mathrm{Lt}_{n=\infty}\sum_0^n a_r\omega_r(x) = f(x).$$

It will be noticed that the success of the above method depended on the subject of integration in the beginning being a geometrical progression. If the general expansion for the unsymmetrical case (33) is treated similarly, the subject of integration is a double geometrical progression and may be treated similarly, but the analysis will not be carried out here.

VII.—*The Application of the ω-functions in the Theory of Probability.*

1. As was mentioned at the outset, the linear ω-functions have been recognised for some time as suitable functions for the purpose of representing a given frequency distribution to any desired degree of approximation. The method commonly adopted in fitting the coefficients is to make the successive moments of the required series equal to those of the given distribution.* But it does not appear to have been noticed anywhere that this method is exactly equivalent to Hermite's method of obtaining the coefficients. This is an immediate consequence of the fact that the function $e^{\frac{1}{2}x^2}u_n$ is a polynomial. Thus the practical method by moments, which commonly appears as an artificial way of obtaining a series to represent a distribution of a particular type, is really the exact and unique way of representing an arbitrary function by a series of ω-functions. Edgeworth† has another method of obtaining a series of successive approximations to a given distribution in the form $\exp\{\sum_{r=s}^{r=t}k_r(-d/dx)^r/r\,!\}\{(2\pi k)^{-\frac{1}{2}}e^{-x^2/2k}\}$, which, if the formal operator be developed, leads to a series of ω-functions. The method of obtaining the coefficients k_r is that of identifying the moments, so that ultimately the series as obtained by Edgeworth must agree with the direct expansion in ω-functions. It might be, however, that, for statistical distributions, the method of development used by Edgeworth would give a

* Charlier, *loc. cit.*

† 'Camb. Phil. Trans.,' Aug., 1904.

more rapid approximation. The argument as to the magnitudes of the coefficients k_r, however, appears equally applicable to the present series. It is shown in the memoir quoted that the coefficients k_r rapidly diminish only if the standard deviation be small. Let this be assumed in forming the series $a_n u_n(x)$, where $u_n = (\partial/\partial x)^n \{(2\pi\sigma^2)^{-\frac{1}{2}} e^{-x^2/2\sigma^2}\}$. Then, if μ_n be the nth moment of the distribution, and $\pi_r = \int_{-\infty}^{\infty} e^{-u^2/2} u^r du$, we have

$$\mu_n = \sum_0^n (-)^r a_r \sigma^{n-r} \pi_{n-r} n!/(n-r)!(2\pi)^{\frac{1}{2}},$$

and in particular $$\mu_2 = (2\pi)^{-\frac{1}{2}} \{a_0 \sigma^2 \pi_2 + 2a_2 \pi_0\},$$

and hence, since σ is the standard deviation, π_2 is zero. Now, following Edgeworth's argument, μ_n is of the order of magnitude σ^n, and hence, by taking the above equations for different values of n, we see that the coefficient a_n is of the order σ^n—the word order being used in the same sense as in the work cited. Thus the coefficients a_n will diminish with about the same rapidity as the quantities k_n, while the formation of the equations is much simpler.

Pass now to the case of two dimensions. Reference has been made to Hermite's note on the expansion of a function in a series of the form

$$\Sigma\Sigma a_{mn} \partial^{m+n}/\partial x^m \partial y^n \, e^{-\frac{1}{2}(ax^2+2hxy+by^2)},$$

which has been shown to be equivalent to an expansion in ω-functions. It is a problem of some importance to obtain a series to represent a frequency distribution in two correlated characteristics, such as, for example, is considered by Perozzo in his memoir on the ages of husband and wife at marriage.* The lines of equal frequency given by Perozzo are far from being concentric ellipses, as they would be if the frequency surface were a Gaussian normal surface $z = e^{-\frac{1}{2}(ax^2+2hxy+by^2)}$. It is therefore worth enquiring whether a few terms of the expansion in ω-functions can be made to give an approximation that is tolerably near. The theoretical possibility of representing the distribution by a convergent series is not here considered, though the conditions of validity of the expansion are almost certainly satisfied. Hermite gives a method of obtaining the coefficients. Using the notation

$$\phi(x) = e^{-\frac{1}{2}(ax^2+2hxy+by^2)}, \qquad \psi(x) = e^{-\frac{1}{2}(bx^2-2hxy+ay^2)},$$

$$(\partial/\partial x)^m (\partial/\partial y)^n \phi = \phi \,.\, \mathrm{U}_{m,n}, \qquad (\partial/\partial x)^m (\partial/\partial y)^n \psi = \psi \mathrm{V}_{mn},$$

$\mathrm{U}_{m,n}$ and V_{mn} are both polynomials of degrees $(m+n)$ in x and y. If $f(x, y)$ be expanded in the form $\Sigma\Sigma a_{m,n} \phi \mathrm{U}_{m,n}$, it is proved that

$$a_{m,n} = \alpha_{m,n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(xy) \, \mathrm{V}_{m,n} dx\, dy,$$

* 'Annali di Statistica,' 1883.

where $\alpha_{m,n}$ is a numerical constant depending on m and n alone. Now it will be much more convenient in practice to obtain the coefficients by means of equating the successive moments $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, x^m y^n dx\, dy$ to the corresponding moments of the series. This is equivalent to Hermite's process, $V_{m,n}$ being a polynomial; so that $a_{m,n}$ above is a linear function of the moments.

Putting $u_{rs} = \phi U_{rs}$, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{rs} x^m y^n dx\, dy$ can be reduced by successive integrations by parts. If $m < r$, we have

$$\iint u_{rs} x^m y^n dx\, dy = (-)^m m! \iint y^n u_{r-m,\,s}\, dx\, dy = 0,$$

whatever the values of s and n, and, similarly, the integral is zero if $n < s$, whatever the values of r and m. But if $m = r$, and $n = s$,

$$\iint u_{mn} x^m y^n dx\, dy = (-)^{m+n}\, m\,!\, n\,! \iint \phi\, dx\, dy,$$

and if $m > r$ and $n > s$,

$$\iint u_{rs} x^m y^n dx\, dy = (-)^{r+s}\, (m\,!/(m-r)\,!)\, (n\,!/(n-s)\,!) \iint \phi x^{m-r} y^{n-s} dx\, dy,$$

and this last integral is again zero if $m-r+n-s$ is an odd integer.

Using the notation $\phi_{rs} = \iint \phi x^r y^s dx\, dy$ and $\mu_{rs} = \iint f x^r y^s dx\, dy$, if we multiply the equation $f(x, y) = \Sigma a_{rs}, u_{rs}$ by $x^r y^s$ and integrate, we have

$$\mu_{rs} = a_{00}\phi_{rs} - (r a_{10}\phi_{r-1,\,s} + s a_{01}\phi_{r,\,s-1})$$
$$+ \{r\,(r-1)\, a_{20}\phi_{r-2,\,s} + r\,.\, s a_{11}\phi_{r-1,\,s-1} + s\,(s-1)\, a_{02}\phi_{r,\,s-2}\}\ldots,$$

there being $(r+1)\,(s+1)$ terms on the right, of which those are zero in which the sum of the subscripts of ϕ is odd.

Taking the first few equations, by choosing the origin at the mean of the distribution, and choosing the constants a, b, h so that

$$\mu_{20} : \mu_{11} : \mu_{02} : \mu_{00} = \phi_{20} : \phi_{11} : \phi_{02} : \phi_{00},$$

the constants a_{01}, a_{10}, a_{20}, a_{11}, a_{02} all become zero. By this means the normal surface that best fits the distribution is obtained, viz., $\mu_{00}\phi/\phi_{00}$.

But the fact that the contour lines in the example quoted are by no means concentric ellipses with centre at the mean shows that the fit is not yet sufficiently good.

The next approximation to the distribution is

$$\mu_{00}\phi/\phi_{00} + (\mu_{30}u_{30} + 3\mu_{21}u_{21} + 3\mu_{12}u_{12} + \mu_{03}u_{03})/6\phi_{00}.$$

The equations giving the coefficients of the fourth order terms are:

$\mu_{40} = a_{00}\phi_{40} + 4!a_{40}\phi_{00}\mu_{31} = a_{00}\phi_{31} + 3!a_{31}\phi_{00}, \quad \mu_{22} = a_{00}\phi_{22} + 2!2!a_{22}\phi_{00}$, etc.

Thus in terms of the moments μ_{rs} the coefficients are quickly obtained, and Edgeworth's argument as to the rapidity of approximation applies equally well here if the standard deviations are sufficiently small. But the labour of obtaining the moments is very great, and a suitable example for testing the closeness of approximation is not available. What will be done here, therefore, is to show more or less generally how, in a simple case, contour lines can be obtained which are approximately of the type found in the example given above (see diagrams, Perozzo, *loc. cit.*).

2. *An Example of Contour Lines in Two Dimensions.*—The example chosen here is that of $\phi = e^{-\frac{1}{2}(x^2+y^2)}$, and it will be assumed that there is symmetry about the line $x = y$.

The most general first correction is $\alpha(u_{30}+u_{03})+\beta(u_{21}+u_{12})$, where $u_{30} = \phi(3x-x^3)$, $u_{21} = \phi y(1-x^2)$, $u_{12} = \phi x(1-y^2)$, $u_{03} = \phi(3y-y^3)$.

In particular, if $\beta = 0$, and α is negative, $\alpha(u_{30}+u_{03})$ vanishes on the straight line $x+y=0$, and the ellipse $x^2-xy+y^2 = 3$; on the positive side of $x+y=0$ it is positive outside the ellipse, and negative inside, and *vice versâ* on the negative side of $x+y=0$.

Thus the effect of the correction $\alpha(u_{30}+u_{03})$, while not altering the mean or second moments, is to introduce *skewness* about $x+y=0$, the maximum ordinate being displaced in the negative direction along $x=y$. Outside the ellipse $x^2-xy+y^2=3$ the surface is depressed on the negative side of $x+y=0$, and raised on the positive side. The zero-line of $\phi+\alpha(u_{30}+u_{03})$ is shown in the figure (curve A) for $\alpha = -1/10$.

The contour lines on the positive part of the surface will now be oval curves surrounding the point of maximum ordinate with their greatest width parallel to $x+y$, and crowding closely together in the neighbourhood of the zero curve. These have not been shown in the figure to avoid complication. A variation in their form could be produced by introducing a term $\beta(u_{12}+u_{21})$, but the effect of skewness is again the most noticeable.

Consider now a fourth order correction, which for the end in view has been conveniently taken to be

$$\gamma(u_{22}-u_{31}-u_{13}) = \gamma\{1-x^2-y^2+6xy-xy(x^2-xy+y^2)\}.$$

The curve for which this is zero is drawn in the figure (curve B), and the portions of the plane are indicated in which it is positive and negative respectively. Thus the effect of this correction on the contour lines in the neighbourhood of the maximum ordinate is to contract them in the direction

$x=-y$, and to elongate them in the direction $x=y$. A series of contour lines is drawn for $\gamma=1/20$; so that the lines shown are curves of equal frequency for the function

$$\{1-\tfrac{1}{10}(\partial^3/\partial x^3+\partial^3/\partial y^3)+\tfrac{1}{20}(\partial^4/\partial x^2\partial y^2-\partial^4/\partial x\partial y^3-\partial^4/\partial y\partial x^3)\}e^{-\frac{1}{2}(x^2+y^2)}.$$

The curves are distinctly similar in type to Perozzo's curves, though no attempt has been made at determining exactly what the coefficients should be. Probably a better approximation would be found with a rather larger coefficient for the third order correction, as the skewness is not so marked as it should be.

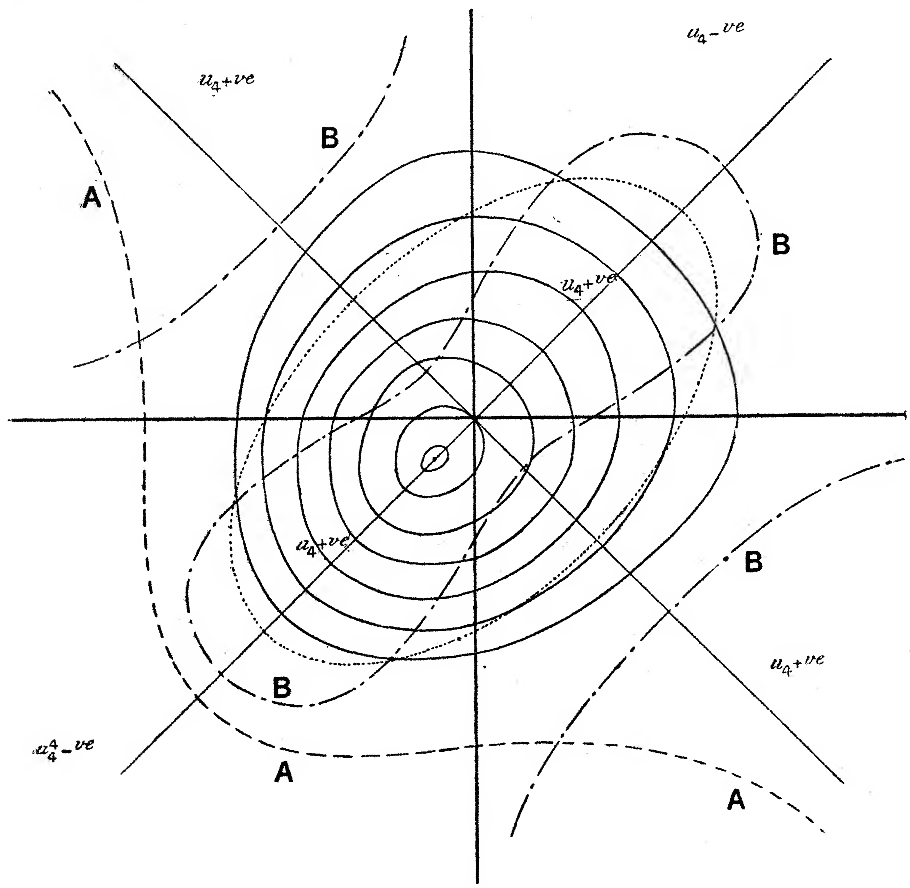

Figure showing Contour Lines of the Surface

$$z=\{1-\tfrac{1}{10}(\partial^3/\partial x^3+\partial^3/\partial y^3)+\tfrac{1}{20}(\partial^4/\partial x^2\partial y^2-\partial^4/\partial x^3\partial y-\partial^4/\partial x\partial y^3)\}e^{-\frac{1}{2}(x^2+y^2)}.$$

A—the curve $\{1-\tfrac{1}{10}(\partial^3/\partial x^3+\partial^3\partial y^3)\}e^{-\frac{1}{2}(x^2+y^2)}=0.$

B—the curve $u_4=\{\partial^4/\partial x^2\partial y^2-\partial^4/\partial x^3\partial y-\partial^4/\partial x\partial y^3\}e^{-\frac{1}{2}(x^2+y^2)}=0.$

Similar curves can clearly be obtained where there is correlation to be taken into account. For by making the change of variable

$$\xi - \eta = \alpha(x-y), \quad \xi + \eta = \beta(x+y), \quad e^{-\frac{1}{2}(x^2+y^2)}$$

is turned into a normal surface with correlation, while $\partial/\partial x$ and $\partial/\partial y$ are linear functions of $\partial/\partial\xi$ and $\partial/\partial\eta$.

Eutectic Research. No. 1.—*The Alloys of Lead and Tin.*

By WALTER ROSENHAIN, B.A., B.C.E., with P. A. TUCKER.

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(Abstract.)

Attempts to prepare pure eutectic alloys of known constitution led to the discovery of discrepancies between the authors' experiments and the data on lead-tin alloys published by Roberts-Austen. A complete redetermination of the equilibria of the lead-tin system was therefore undertaken, by both pyrometric and microscopical methods. Cooling-curves of the alloys taken by both inverse-rate and differential methods are given; these, together with the microscopic data, lead to the equilibrium diagram shown in the figure. This differs principally from that given by Roberts-Austen,* in that the eutectic point is placed at a concentration of 63 per cent. of tin instead of 69, that the eutectic line towards the lead end of the series terminates at a concentration close to 16 per cent. of tin, and that a series of transformations (along the line EFG) have been found in the solid alloys near the lead end of the series.

The discrepancies of these results, as regards the eutectic composition, arise from the more delicate method employed in the present research. While alloys within 1 per cent. on either side of the true eutectic composition show no detectable difference in freezing- or melting-point, the presence of small amounts of excess of either constituent can be detected by the microscope, and in this manner the composition of the pure eutectic has been ascertained. As regards the solubility of tin in solid lead, it was found that the occurrence of the eutectic arrest-point in alloys containing less than 16 per cent. of tin

* Roberts-Austen, 'Fifth Report to the Alloys Research Committee of the Institutio of Mechanical Engineers,' 1897.